

# Quantization of Dirac fields in static spacetime

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**Abstract.** On a static spacetime, the solutions of Dirac equation are generated by a time-independent Hamiltonian. We study this Hamiltonian and characterize the split into positive and negative energy. We use it to find explicit expressions for advanced and retarded fundamental solutions and for the propagator. Finally we use a fermion Fock space based on the positive/negative energy split to define a Dirac quantum field operator whose commutator is the propagator.

# 1. Introduction

In the theoretical framework of Dirac fields in curved space-time, many fundamental results have been obtained by Lichnorowicz [1]. There are also quite a few standard references with detailed discussions, for example, see [2][3][4]. A very general theory has been established by Dimock [5], for Dirac quantum fields on hyperbolic Lorentzian manifold. For most up-to-date research along this line, one can find recent papers by Hollands [6] and Kratzert [7], and references therein.

Here in this paper we focus on solving a very specific and well-posed problem: the quantization of Dirac fields in static space-time. Technically speaking, the separation of time and space parts of Dirac fields plays a pivotal role in doing field quantization. Since only static space-time is considered here, the following results about other type of fields are also of relevance: the quantization of Klein-Gordon scalar fields on stationary manifold by Kay [8], and the quantization of electromagnetic fields and massive vector fields on static space-time by Furlani [9][10].

This paper proceeds as follows. First we present some preliminary results for classical Dirac fields. In static space-time, the time component of the spin affine connection of Dirac fields vanishes and the other components are all independent of time. Hence one can separate space and time. The dynamics can be expressed in terms of a time-independent Hamiltonian. We prove that the Hamiltonian is essentially self-adjoint. We also characterize the positive and negative energy subspaces. This leads to explicit expressions for various fundamental solutions and for the propagator function.

For the quantum problem, we first define an appropriate fermion Fock space based on the positive/negative energy split. Then Dirac quantum field operators are defined using the creation and annihilation operators on this Fock space and projections onto the positive and negative energy subspaces. The field operator is shown to have a commutator which is the propagator function. Finally we study the unitary implementability of the time evolution.

## 2. Dirac fields in static space-time

In this section, we present some preliminary results that are relevant to our work. On Lorentzian manifold  $L$ , the first-order equation of free Dirac fields can be written as

$$(\mathcal{P} - m)\psi = 0, \quad (2.1a)$$

$$\psi^*(\mathcal{P}^* - m) = 0, \quad (2.1b)$$

where, by the notation in [11][12],  $\psi^* = \psi^\dagger \gamma^0$  and  $\mathcal{P}^* = \gamma^0 \mathcal{P}^\dagger \gamma^0$  and

$$\mathcal{P} = i\gamma^\mu \nabla_\mu = i\gamma^\mu (\partial_\mu - \Gamma_\mu). \quad (2.2)$$

Here  $\gamma^\mu = V_a^\mu(x)\gamma^a$  are spinor tensors, with the introduction of vierbein fields  $V_a^\mu(x)$  and Dirac matrices  $\gamma^a$  ( $a = 0, 1, 2, 3$ ) by the convention in [13]. The components of spin affine connection are

$$\Gamma_\mu = \frac{1}{2}G^{[a,b]}(\mathcal{D}_\mu V_a^\nu)V_{\nu b}, \quad (2.3)$$

where  $G^{[a,b]} = \frac{1}{4}[\gamma^a, \gamma^b]$  are the generators of Lorentz group, and  $\mathcal{D}_\mu V_a^\nu = \partial_\mu V_a^\nu + \Gamma_{\mu\lambda}^\nu V_a^\lambda$  are the covariant derivatives of vierbein fields on space.

The second-order equation of Dirac fields can be written as

$$(\square - m^2)\psi = 0, \quad (2.4)$$

with an operator

$$\square = \not{p}^2 = -\nabla^\mu \nabla_\mu + \frac{1}{4}R, \quad (2.5)$$

where  $R$  is the Riemann scalar.

To preserve manifest covariance on Lorentzian Manifold  $L$ , we may introduce an indefinite inner product [1]:

$$\langle u, v \rangle = \int u^*(x)v(x)d_g^4x, \quad u, v \in C^\infty(L; C^4), \quad (2.6)$$

where  $d_g^4x = \sqrt{g}d^4x$  is the invariant density with  $g = -\det(g^{\mu\nu})$  and  $u^* = u^\dagger \gamma^0$  is the adjoint of  $u$ . The inner product is invariant under both global coordinate and local Lorentz transformations.

The adjoint operator  $A^*$  of  $A$  is defined by

$$\langle u, A^*v \rangle = \langle Au, v \rangle, \quad (2.7)$$

namely

$$A^* = \gamma^0 A^\dagger \gamma^0. \quad (2.8)$$

A symmetric operator  $O$  satisfies

$$\langle u, Ov \rangle = \langle Ou, v \rangle, \quad (2.9)$$

namely

$$O^* = \gamma^0 O^\dagger \gamma^0 = O. \quad (2.10)$$

Symmetric operators play essential roles in functional analysis.

Let us consider a static manifold  $R \times M$  where  $M$  is compact, with metric elements  $g^{\mu\nu}$  of signature  $(1, -1, -1, -1)$ ,

$$[g^{\mu\nu}] = \begin{bmatrix} 1 & 0 \\ 0 & g^{ij}(\mathbf{x}) \end{bmatrix}, \quad (i, j = 1, 2, 3) \quad (2.11)$$

where Greek indices apply to 4-d static space-time, and Latin indices apply to 3-d static space. The 3-d static space  $M$  is a Cauchy surface of the 4-d static space-time  $R \times M$ .

Let  $u$  and  $v$  have compact support. Then check Green's identity in static space-time  $L = R \times M$  where  $M$  is compact without boundary

$$\begin{aligned} \langle \not{p}u, v \rangle - \langle u, \not{p}v \rangle &= \int_L d_g^4x [(\not{p}u)^*v - u^*\not{p}v] \\ &= -i \int_L d_g^4x [(\nabla_\nu u)^* \gamma^\nu v + u^* \gamma^\mu \nabla_\mu v] \\ &= -i \int_L d_g^4x \nabla_\mu (u^* \gamma^\mu v), \end{aligned} \quad (2.12)$$

where  $\nabla_\mu \gamma^\nu = 0$  has been used. This integral is the same as the integral over  $[-T, T] \times M$  for  $T$  sufficiently large depending on the test functions. The integral of the divergence is

an integral over the boundary of this region which is  $-T \times M$  and  $T \times M$ . The surface integrals are zero. We therefore have

$$\langle \mathcal{P}u, v \rangle = \langle u, \mathcal{P}v \rangle, \quad (2.13)$$

namely the operator  $\mathcal{P}$  is symmetric

$$\mathcal{P}^* = \mathcal{P}. \quad (2.14)$$

Obviously  $\Delta = \mathcal{P}^2$  is also symmetric by the same arguments as for  $\mathcal{P}$ .

The vierbein fields satisfy  $g_{\mu\nu}(x)V_a^\mu(x)V_b^\nu(x) = \eta_{ab}$  with a Minkowski metric  $\{\eta_{ab}\} = \text{diag}(1, -1, -1, -1)$ . In static space-time, the vierbein fields  $V_0^0 = 1$ ,  $V_i^0 = 0 = V_0^i$  and  $V_a^i(\mathbf{x})$  ( $i, a = 1, 2, 3$ ) are all independent of time. Then by (2.3) it is easy to show that the time component of the spin affine connection vanishes:

$$\begin{aligned} \Gamma_{0\lambda}^\nu &= \frac{1}{2}g^{\nu\sigma}(\partial_\lambda g_{\sigma 0} + \partial_0 g_{\sigma\lambda} - \partial_\sigma g_{0\lambda}) = 0, \\ \mathcal{D}_0 V_a^\nu &= \partial_0 V_a^\nu + \Gamma_{0\lambda}^\nu V_a^\lambda = 0, \\ \Gamma_0 &= \frac{1}{2}G^{[a,b]}(\mathcal{D}_0 V_a^\nu)V_{\nu b} = 0, \end{aligned} \quad (2.15)$$

and also the other components  $\Gamma_i(\mathbf{x})$  ( $i = 1, 2, 3$ ) are all independent of time.

Separating time from space, we assume there exists a global dreibein field on  $M$ . If there is not a dreibein field, our analysis should still hold true, however the spinor fields will be sections of a vector bundle [5]. The simplest example of a compact manifold  $M$  with a global dreibein field is  $M = T^3$ , the three torus or periodic box.

### 3. Separation of energy spectrum

With the above results in static space-time, the Dirac equation (2.1a) turns out to be

$$i\partial_t \psi(t, \mathbf{x}) = H\psi(t, \mathbf{x}), \quad (3.1)$$

where a time-independent Hamiltonian is

$$H = -i\gamma^0 \gamma^i \nabla_i + \gamma^0 m. \quad (3.2)$$

And the Hamiltonian squared is

$$H^2 = \nabla^i \nabla_i - \frac{1}{4}R + m^2. \quad (3.3)$$

Since we know  $\gamma^i(\mathbf{x}) = V_a^i(\mathbf{x})\gamma^a$  and  $-\gamma^0 \gamma^a = \sigma^1 \otimes \sigma^a$  where  $\sigma^a$  are Pauli matrices. The Hamiltonian can then be written in a matrix form

$$H = \begin{bmatrix} m & Q \\ Q & -m \end{bmatrix}, \quad (3.4)$$

where we denote  $Q = i\sigma^i \nabla_i$  and  $\sigma^i = V_a^i(\mathbf{x})\sigma^a$  ( $i, a = 1, 2, 3$ ). The Hamiltonian squared becomes

$$H^2 = (m^2 + Q^2) \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad (3.5)$$

where  $Q^2 = \nabla^i \nabla_i - \frac{1}{4}R$ .

Define a positive-definite inner product on  $M$

$$(\chi, \varphi) = \int d_g^3 \mathbf{x} (\chi^\dagger \varphi), \quad (3.6)$$

where  $d_g^3 \mathbf{x} = \sqrt{g} d^3 \mathbf{x}$ . Then define a Hilbert space

$$\mathcal{H} = L^2(M, C^4, d_g^3 \mathbf{x}). \quad (3.7)$$

Since the vierbein fields  $V_a^i(\mathbf{x})$  are real functions of space on  $M$  and Pauli matrices  $\sigma^a$  are symmetric on  $C^2$ ,  $\sigma^i = V_a^i(\mathbf{x})\sigma^a$  are thus symmetric in  $L^2(M, C^2)$ . It can be easily checked that  $G^{[a,b]\dagger} = -G^{[a,b]}$ . By (2.3), we know  $(i\Gamma_i)^\dagger = i\Gamma_i$  and  $i\Gamma_i$  are some real functions of space on  $M$  after summation of indices. So  $i\nabla_i = i(\partial_i - \Gamma_i)$  is symmetric in  $L^2(M)$ , and  $i\sigma^i \nabla_i$  is symmetric in  $L^2(M, C^2)$ . Then we know the Hamiltonian  $H$  is symmetric in  $L^2(M, C^4)$ .

Let  $A$  be the closure of  $H$  on  $C^\infty(M, C^4)$  in  $L^2(M, C^4)$ . Define the domain of  $A$  as follows

$$D(A) = \left\{ \psi \in \mathcal{H} : \exists \psi_n \in C^\infty M, \lim_{n \rightarrow \infty} \psi_n \rightarrow \psi, \lim_{n \rightarrow \infty} A\psi_n \text{ exists} \right\}. \quad (3.8)$$

Then  $A : D(A) \rightarrow \mathcal{H}$  is given by defining

$$A\psi = \lim_{n \rightarrow \infty} A\psi_n. \quad (3.9)$$

Let  $B = A^2$  be the closure of  $H^2$  on  $C^\infty(M, C^4)$ .

*Lemma 1.*  $B$  is self-adjoint in  $\mathcal{H}$ , i.e.  $H^2$  is essentially self-adjoint on  $C^\infty(M, C^4)$ .

*Proof:* It has been proven that the Laplacian operator  $-\nabla^i \nabla_i$  is self-adjoint in  $\mathcal{H}$  [14][15]. Since  $-\frac{1}{4}R + m^2$  is a continuous function on a compact manifold, it is a bounded function and hence a bounded operator in  $\mathcal{H}$  [16][17]. By Kato-Rellich theorem [18],  $B$ , a closure of  $H^2 = \nabla^i \nabla_i - \frac{1}{4}R + m^2$ , is self-adjoint in  $\mathcal{H}$ . This is equivalent to say  $H^2$  is essentially self-adjoint on  $C^\infty(M, C^4)$ .  $\square$

*Lemma 2.*  $A$  is self-adjoint in  $\mathcal{H}$ , i.e.  $H$  is essentially self-adjoint on  $C^\infty(M, C^4)$ .

*Proof:* This is equivalent to show [16]

$$\text{Ran}(A \pm i) = \mathcal{H}. \quad (3.10)$$

To prove it, we need to find  $\psi$  so that

$$(A \pm i)\psi = \chi, \quad \text{for } \chi \in \mathcal{H}. \quad (3.11)$$

By observation, the answer should be

$$\psi = (A \mp i)(B + 1)^{-1} \chi. \quad (3.12)$$

It then suffices to show

$$\varphi = (B + 1)^{-1} \chi \in D(A), \quad (3.13a)$$

$$\psi = (A \mp i)\varphi \in D(A). \quad (3.13b)$$

By *Lemma 1*, for any  $\varphi \in D(B)$  we can find smooth  $\varphi_n$  so that

$$\lim_{n \rightarrow \infty} \varphi_n \rightarrow \varphi, \quad \lim_{n \rightarrow \infty} B\varphi_n \rightarrow B\varphi. \quad (3.14)$$

Now we derive

$$\begin{aligned} \|A\varphi_n - A\varphi_m\|^2 &= (\varphi_n - \varphi_m, A^2(\varphi_n - \varphi_m)) \\ &\leq \|\varphi_n - \varphi_m\| \times \|B(\varphi_n - \varphi_m)\| \rightarrow 0. \end{aligned} \quad (3.15)$$

So we know  $\varphi \in D(A)$  by (3.8). Then we may define smooth  $\psi_n = (A \mp i)\varphi_n$  so that  $\psi_n \rightarrow \psi$  and derive

$$\begin{aligned} \|A\psi_n - A\psi_m\|^2 &= (A^2(\varphi_n - \varphi_m), (A^2 + 1)(\varphi_n - \varphi_m)) \\ &\leq \|B(\varphi_n - \varphi_m)\| \times \|(B + 1)(\varphi_n - \varphi_m)\| \rightarrow 0. \end{aligned} \quad (3.16)$$

Thus we know  $\psi \in D(A)$ . By the above analysis,  $A$  is self-adjoint in  $\mathcal{H}$ , and equivalently  $H$  is essentially self-adjoint on  $C^\infty(M, C^4)$ .  $\square$

By *Lemma 1*,  $H^2$  is essentially self-adjoint and positive, and hence has a square root. We may define a positive scalar energy operator

$$\omega = (m^2 + Q^2)^{1/2} = (\nabla^i \nabla_i - \frac{1}{4}R + m^2)^{1/2}, \quad (3.17)$$

where  $Q = i\sigma^i \nabla_i$  for  $\sigma^i = V_a^i(\mathbf{x})\sigma^a$ . By *Lemma 2*, the closure of  $H$  is self-adjoint, and similarly the closure of  $Q$  is self-adjoint, then  $Q^2 = \nabla^i \nabla_i - R/4 \geq 0$  and  $\omega \geq m$ . It is now straightforward to prove the following Theorem.

*Theorem 1:* The Hilbert space  $\mathcal{H}$  splits into the positive and negative subspaces:  $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ . Then for  $\psi_\pm \in \mathcal{H}^\pm \cap D(H)$ , we have

$$H\psi_\pm = \pm\omega\psi_\pm, \quad (3.18)$$

where  $\omega$  is expressed by (3.17). And also  $\psi_\pm$  are of the following form:

$$\psi_+ = T \begin{bmatrix} f \\ 0 \end{bmatrix}, \quad \psi_- = T \begin{bmatrix} 0 \\ h \end{bmatrix}, \quad f, h \in L^2(M; C^2), \quad (3.19)$$

where  $T$  is the unitary operator

$$T = N \begin{bmatrix} \omega + m & -Q \\ Q & \omega + m \end{bmatrix}, \quad (3.20)$$

with  $N = [2\omega(\omega + m)]^{-1/2}$ .

*Proof:* We diagonalize the Hamiltonian (3.4) to

$$H' = T^{-1}HT = \begin{bmatrix} \omega & 0 \\ 0 & -\omega \end{bmatrix}, \quad (3.21)$$

by using a transformation operator (3.20). The inverse of  $T$  is

$$T^{-1} = N \begin{bmatrix} \omega + m & Q \\ -Q & \omega + m \end{bmatrix}. \quad (3.22)$$

Both  $T$  and  $T^{-1}$  are norm-preserving:

$$\int (T\varphi)^\dagger T\varphi d\mu = \int \varphi^\dagger \varphi d\mu = \int (T^{-1}\varphi)^\dagger T^{-1}\varphi d\mu, \quad (3.23)$$

so  $T$  is unitary.

Any  $\psi$  in  $\mathcal{H}$  can be written as

$$\psi = T \begin{bmatrix} f \\ h \end{bmatrix}, \quad f, h \in L^2(M; C^2) \quad (3.24)$$

and hence  $\psi = \psi_+ + \psi_-$  where

$$\psi_+ = T \begin{bmatrix} f \\ 0 \end{bmatrix}, \quad \psi_- = T \begin{bmatrix} 0 \\ h \end{bmatrix}, \quad f, h \in L^2(M; C^2). \quad (3.25)$$

If also  $\psi_\pm \in D(H)$ , then

$$H\psi_\pm = \pm\omega\psi_\pm. \quad (3.26)$$

Thus we have exhibited the split into the positive and negative energy spectra. (Note that 0 is not in the spectra since  $\omega \geq m > 0$ .)  $\square$

By the above *Theorem 1*, we can now express Dirac fields by summing up both positive and negative energy parts

$$\psi(t, \mathbf{x}) = U(t)\psi_+(\mathbf{x}) + U(-t)\psi_-(\mathbf{x}), \quad (3.27)$$

which is a solution of Dirac equation with data  $\psi$ . Here  $\psi_\pm$  satisfy (3.25), and a unitary operator  $U(t) = \exp(-i\omega t)$  satisfies  $U^\dagger(t) = U(-t) = U^{-1}(t)$  with  $\omega$  given by (3.17).

## 4. Propagator of Dirac equation

To obtain the propagator of Dirac equation, we start from the second-order inhomogeneous equation

$$(\square - m^2)\psi = \rho, \quad (4.1)$$

where  $\square$  is expressed by (2.5). The fundamental solutions of this inhomogeneous equation are defined by

$$(\square_x - m^2)E_F(x, y) = \delta(x, y). \quad (4.2)$$

Here the  $\delta$ -function is defined by a bispinor [1]

$$\delta(x, y) = \delta_\beta^\alpha \delta^4(x, y). \quad (4.3)$$

The general discussions about the existence and uniqueness of fundamental solutions of hyperbolic differential equations such as (4.1) can be found in [3][4]. Now we are going to do a formal calculation to find a representation for the advanced and retarded fundamental solutions  $E_A, E_R$ .

On static metric  $R \times M$ , we separate the time and space parts of the solutions. It is translation-invariant along time direction but not necessarily along space direction. We make Fourier transform along time axis but leave the space part alone. The fundamental solutions can be written as

$$E_F(x, y) = \frac{1}{2\pi} \int e^{-ikt} E_k(\mathbf{x}, \mathbf{y}) dk, \quad (4.4)$$

where

$$t = x_0 - y_0. \quad (4.5)$$

Inserting (4.4) into (4.2) leads to

$$(k^2 - \nabla^i \nabla_i + \frac{1}{4}R - m^2)E_k(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x}, \mathbf{y}). \quad (4.6)$$

We formally write k-component solution

$$E_k(\mathbf{x}, \mathbf{y}) = (k^2 - \omega^2)^{-1} \delta(\mathbf{x}, \mathbf{y}), \quad (4.7)$$

with  $\omega^2 = \nabla^i \nabla_i - R/4 + m^2$ . So (4.4) formally becomes

$$E_F(x, y) = \frac{1}{2\pi} \int \frac{e^{-ikt}}{k^2 - \omega^2} \delta(\mathbf{x}, \mathbf{y}) dk, \quad (4.8)$$

which is singular with a delta function. Let us smear it by two test functions  $\chi(\mathbf{x})$  and  $\varphi(\mathbf{y})$  on  $M$

$$E_F(t; \chi, \varphi) = \frac{1}{2\pi} \int e^{-ikt} (\chi, \frac{1}{k^2 - \omega^2} \varphi) dk. \quad (4.9)$$

Let  $P_\lambda = P_{(-\infty, \lambda]}$  be a projection-valued measure of self-adjoint operator  $\omega$ . Its family  $\{P_\lambda\}$  exists by the Spectral Theorem [16]. Thus (4.9) becomes<sup>1</sup>

$$E_F(t; \chi, \varphi) = \frac{1}{2\pi} \int \int \frac{e^{-ikt}}{k^2 - \lambda^2} dk (\chi, dP_\lambda \varphi), \quad \forall \chi, \varphi \in \mathcal{H}. \quad (4.10)$$

We may now prove the following proposition.

*Proposition 1:* The advanced and retarded fundamental solutions in integral representation, defined by

$$E_A(t; \chi, \varphi) = \frac{1}{2\pi} \int \int_{\Gamma_A} \frac{e^{-ikt}}{k^2 - \lambda^2} dk (\chi, dP_\lambda \varphi), \quad \forall \chi, \varphi \in \mathcal{H}, \quad (4.11a)$$

$$E_R(t; \chi, \varphi) = \frac{1}{2\pi} \int \int_{\Gamma_R} \frac{e^{-ikt}}{k^2 - \lambda^2} dk (\chi, dP_\lambda \varphi), \quad \forall \chi, \varphi \in \mathcal{H}, \quad (4.11b)$$

vanish in the future and past respectively. Here  $\Gamma_A$  ( $\Gamma_R$ ) is a straight line slightly below (above) the real k-axis.

*Proof:* Obviously (4.10) has two poles  $k = \pm \lambda$  for each  $\lambda$  in the integral over  $k$ . Let us avoid two poles in (4.11a) by taking the integral along a straight line  $\Gamma_A$  which passes slightly below the real k-axis. This is equivalent to moving the poles to slightly above the real k-axis. When  $t > 0$ , we calculate the integral by closing the contour in the lower half of the complex k-plane. Since  $Re(-ikt) < 0$ , the infinite semicircle boundary in the lower half plane does not contribute. And there is no pole inside the closed contour,  $E_A$  vanishes in the future when  $t > 0$ . Similarly  $E_R$  defined by (4.11b) vanishes in the past when  $t < 0$ , by the integral along a straight line  $\Gamma_R$  which passes slightly above the real k-axis. It is also straightforward to check that the expressions are actually fundamental

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<sup>1</sup>Since we are considering compact manifolds, the spectral integral is actually a sum over discrete eigenvalues. Nevertheless we stick with the more general notation.



solutions. From [3][4], the fundamental solutions of hyperbolic differential equations have support only in light cone. So the advanced and retarded fundamental solutions  $E_A$  and  $E_R$  have support only in the past and future light cones respectively.  $\square$

Let us take the difference of (4.11a) and (4.11b) by closing contour  $C$  around both poles

$$\begin{aligned} E(t; \chi, \varphi) &= E_A(t; \chi, \varphi) - E_R(t; \chi, \varphi) \\ &= \frac{1}{2\pi} \int \oint_C \frac{e^{-ikt}}{k^2 - \lambda^2} dk(\chi, dP_\lambda \varphi). \end{aligned} \quad (4.12)$$

It can also be split into the positive and negative parts by choosing the contours around both poles:

$$E(t; \chi, \varphi) = E_+(t; \chi, \varphi) + E_-(t; \chi, \varphi), \quad (4.13)$$

where

$$\begin{aligned} E_\pm(t; \chi, \varphi) &= \frac{1}{2\pi} \int \oint_{C_\pm} \frac{e^{-ikt}}{k^2 - \lambda^2} dk(\chi, dP_\lambda \varphi) \\ &= \pm i \int \frac{e^{\mp i\lambda t}}{2\lambda} (\chi, dP_\lambda \varphi) \\ &= \pm i (\chi, \frac{e^{\mp i\omega t}}{2\omega} \varphi). \end{aligned} \quad (4.14)$$

The fundamental solutions of the inhomogeneous Dirac equation

$$(\mathbb{P} - m)\psi = \rho, \quad (4.15)$$

are defined by

$$(\mathbb{P} - m)S_F(x, y) = \delta(x, y). \quad (4.16)$$

From  $(\mathbb{P} - m)(\mathbb{P} + m) = \square - m^2$ , we see

$$S_F(x, y) = (\mathbb{P} + m)E_F(x, y), \quad (4.17)$$

which can be smeared by two test functions  $\chi(\mathbf{x})$  and  $\varphi(\mathbf{y})$  on  $M$ :

$$S_F(t; \chi, \varphi) = \frac{1}{2\pi} \int e^{-ikt} dk(\chi, \frac{\mathbb{P}(k) + m}{k^2 - \omega^2} \varphi), \quad (4.18)$$

where the operator  $\mathbb{P}(k)$  is also a function of  $k$ :

$$\mathbb{P}(k) = \gamma^0 k + i\gamma^i \nabla_i. \quad (4.19)$$

Following the discussions about  $E_\pm$  (4.14), we obtain

$$\begin{aligned} S_\pm(t; \chi, \varphi) &= \frac{1}{2\pi} \int \oint_{C_\pm} e^{-ikt} dk(\chi, \frac{\mathbb{P}(k) + m}{k^2 - \lambda^2} dP_\lambda \varphi) \\ &= \pm i \int e^{\mp i\lambda t} (\chi, \frac{\mathbb{P}(\pm\lambda) + m}{2\lambda} dP_\lambda \varphi) \\ &= (\chi, i e^{\mp i\omega t} \pi_\pm \gamma^0 \varphi). \end{aligned} \quad (4.20)$$

Here  $\pi_{\pm}$  are two orthogonal projection operators onto the positive and negative energy parts in Hilbert space respectively,

$$\pi_{\pm} = \frac{\omega \pm (-i\gamma^0\gamma^i\nabla_i + \gamma^0m)}{2\omega}. \quad (4.21)$$

We can check they are the correct projection operators by applying them on the positive and negative energy solutions (3.19):

$$\pi_{\pm}\psi_{\pm} = \psi_{\pm}, \quad \pi_{\pm}\psi_{\mp} = 0. \quad (4.22)$$

Generally,  $\pi_{\pm}$  have the following relations:

$$\pi_{\pm}^{\dagger} = \pi_{\pm} = \pi_{\pm}^2, \quad (4.23a)$$

$$\pi_{\pm}\pi_{\mp} = 0, \quad (4.23b)$$

$$\pi_{+} + \pi_{-} = 1. \quad (4.23c)$$

And  $\pi_{\pm}$  commute with  $\omega$ , since  $\omega^2$  commutes with  $\gamma^i\nabla_i$  and  $\omega$  is a scalar.

By (4.20), the positive and negative energy parts of  $S$ -function in space-time representation can be formally expressed by

$$S_{\pm}(x, y) = ie^{\mp i\omega t}\pi_{\pm}(\mathbf{x})\gamma^0\delta(\mathbf{x}, \mathbf{y}), \quad (4.24)$$

which make sense mathematically only if they are smeared as in (4.20).

Similar to  $E(x, y)$ , the propagator of Dirac equation is defined as the difference between the advanced and retarded fundamental solutions:

$$S(x, y) = S_A(x, y) - S_R(x, y). \quad (4.25)$$

where  $S_A, S_R$  are related to  $E_A, E_R$  by (4.17), or the summation of the positive and negative energy fundamental solutions:

$$S(x, y) = S_{+}(x, y) + S_{-}(x, y). \quad (4.26)$$

Since  $\text{supp}(S_R u) \cap \text{supp}(S_A v)$  is compact, by using the same procedure of deriving equation (2.13) we compute

$$\begin{aligned} &< S_R u, v > = < S_R u, (\mathcal{P} - m)S_A v > \\ &= < (\mathcal{P} - m)S_R u, S_A v > = < u, S_A v > . \end{aligned} \quad (4.27)$$

Here we can see the advanced and retarded fundamental solutions are the adjoints of each other in spin space

$$[S_R(x, y)]^* = S_A(y, x), \quad (4.28)$$

and similarly

$$[S_{\pm}(x, y)]^* = -S_{\pm}(y, x). \quad (4.29)$$

Therefore the propagator satisfies

$$[S(x, y)]^* = -S(y, x), \quad (4.30)$$

and there is an obvious result

$$[-iS(x, y)]^* = -iS(y, x). \quad (4.31)$$

This completes our discussions on the propagator of Dirac equation.

## 5. Quantization of Dirac fields

To construct field theory one needs to define a Fock space with one particle space as its base space [19]. A general Fock space is defined by [20][21]

$$\mathcal{F}(\mathcal{H}) = \oplus_{n=0}^{\infty} \mathcal{F}^{(n)}(\mathcal{H}) = \mathcal{F}^{(0)}(\mathcal{H}) \oplus \dots \oplus \mathcal{F}^{(n)}(\mathcal{H}) \oplus \dots, \quad (5.1)$$

where n-fold tensor subspace is

$$\mathcal{F}^{(n)}(\mathcal{H}) = \otimes \mathcal{H}^{(n)} = \mathcal{H} \otimes \dots \otimes \mathcal{H}, \quad (5.2)$$

and  $\mathcal{F}^{(0)}(\mathcal{H}) = C$  is the vacuum space with complex constants as its elements.  $\mathcal{H}$  is any complex Hilbert space with a positive-definite inner product. In Fock space  $\psi \in \mathcal{F}(\mathcal{H})$

$$\psi = (\psi^{(0)}, \psi^{(1)}, \dots, \psi^{(n)}, \dots), \quad (5.3)$$

where  $\psi^{(n)} \in \mathcal{F}^{(n)}(\mathcal{H})$ . A dense set in  $\mathcal{F}^{(n)}(\mathcal{H})$  is linear combinations of vectors of the form

$$\psi^{(n)} = \psi_1 \otimes \dots \otimes \psi_n. \quad (5.4)$$

The inner product on  $\mathcal{F}(\mathcal{H})$  is induced by the inner product on  $\mathcal{H}$ :

$$(\psi, \psi)_{\mathcal{F}} = |\psi^{(0)}|^2 + (\psi^{(1)}, \psi^{(1)})_{\mathcal{H}^{(1)}} + \dots + (\psi^{(n)}, \psi^{(n)})_{\mathcal{H}^{(n)}} + \dots \quad (5.5)$$

If  $\psi^{(n)}$  has the form (5.4), then

$$(\psi^{(n)}, \psi^{(n)})_{\mathcal{H}^{(n)}} = \prod_{i=1}^n (\psi_i, \psi_i)_{\mathcal{H}}. \quad (5.6)$$

The construction of Fock space can be put in the above form for both fermion and boson fields. Here we are only interested in Dirac fermion fields which obey antisymmetric rule. Define linear permutation operators on  $\mathcal{F}(\mathcal{H})$  by

$$\Pi(\psi_1 \otimes \dots \otimes \psi_n) = \frac{1}{n!} \sum_{\pi} (-1)^{\pi} \psi_{\pi(1)} \otimes \dots \otimes \psi_{\pi(n)}, \quad (5.7)$$

which induces orthogonal projections onto  $\mathcal{F}^{(n)}(\mathcal{H})$ . For example,

$$\Pi(\psi_1 \otimes \psi_2) = \frac{1}{2}(\psi_1 \otimes \psi_2 - \psi_2 \otimes \psi_1). \quad (5.8)$$

Applying this operator to  $\mathcal{F}(\mathcal{H})$ , we obtain antisymmetric fermionic Fock space

$$\mathcal{F}_a(\mathcal{H}) = \oplus_{n=0}^{\infty} \mathcal{F}_a^{(n)}(\mathcal{H}), \quad (5.9)$$

in which n-fold subspaces are defined by

$$\mathcal{F}_a^{(n)}(\mathcal{H}) = \Pi \mathcal{F}^{(n)}(\mathcal{H}). \quad (5.10)$$

To have a complete description of Fock space, one should define creation and annihilation operators. On the vacuum space  $\mathcal{F}^{(0)}(\mathcal{H})$ , one defines

$$a_0(\chi)\psi_0 = 0, \quad (5.11a)$$

$$a_0^\dagger(\chi)\psi_0 = \chi. \quad (5.11b)$$

Generally, one can define the creation and annihilation operators on  $\mathcal{F}(\mathcal{H})$  by

$$a_0^\dagger(\chi)(\psi_1 \otimes \dots \otimes \psi_n) = \sqrt{n+1}(\chi \otimes \psi_1 \otimes \dots \otimes \psi_n), \quad (5.12a)$$

$$a_0(\chi)(\psi_1 \otimes \dots \otimes \psi_n) = \sqrt{n}(\chi, \psi_1)_{\mathcal{H}}(\psi_2 \otimes \dots \otimes \psi_n). \quad (5.12b)$$

Then on Fermi-Fock space  $\mathcal{F}_a(\mathcal{H})$ , according to Bratteli and Robinson [22], the creation operator  $a^\dagger(\chi)$  and annihilation operator  $a(\chi)$  can be defined as

$$a^\dagger(\chi) = \Pi a_0^\dagger(\chi) \Pi, \quad (5.13a)$$

$$a(\chi) = \Pi a_0(\chi) \Pi. \quad (5.13b)$$

It is straightforward to show the creation and annihilation operators satisfy the following canonical anticommutation relations (CAR)

$$\{a(\chi), a(\varphi)\} = 0, \quad (5.14a)$$

$$\{a^\dagger(\chi), a^\dagger(\varphi)\} = 0, \quad (5.14b)$$

$$\{a(\chi), a^\dagger(\varphi)\} = (\chi, \varphi)_{\mathcal{H}}, \quad (5.14c)$$

where  $(\chi, \varphi)_{\mathcal{H}}$  represents an inner product in Hilbert space.

Let us define a Fermi-Fock space in static space-time

$$\mathcal{F}_a(\mathcal{H}) = \mathcal{F}_a(\mathcal{H}_+) \otimes \mathcal{F}_a(\mathcal{H}_-), \quad (5.15)$$

where  $\mathcal{F}_a(\mathcal{H}_\pm)$  are the positive and negative subspaces respectively. Also define

$$a_+(\chi) = a(\chi) \otimes I, \quad a_+^\dagger(\chi) = a^\dagger(\chi) \otimes I; \quad (5.16a)$$

$$a_-(\chi) = (-1)^N \otimes a(\chi), \quad a_-^\dagger(\chi) = (-1)^N \otimes a^\dagger(\chi), \quad (5.16b)$$

where  $(-1)^N$  is necessary so that  $a_+$  and  $a_-$  anti-commute [20], and  $a(\chi), a^\dagger(\chi)$  are defined in (5.13) with inner product  $(\chi, \varphi)_{\mathcal{H}} = (\chi, \varphi)$ . It is straightforward to check that the so-defined annihilation and creation operators satisfy the following CAR

$$\{a_\pm(f_\pm), a_\pm(h_\pm)\} = 0, \quad (5.17a)$$

$$\{a_\pm^\dagger(f_\pm), a_\pm^\dagger(h_\pm)\} = 0, \quad (5.17b)$$

$$\{a_\pm(f_\pm), a_\pm^\dagger(h_\pm)\} = (f_\pm, h_\pm), \quad (5.17c)$$

where  $f_\pm$  are any vectors in the positive and negative energy subspaces respectively.

The classical solutions of the Dirac equation are given by (3.27). Correspondingly we define a quantum field operator to be the solution with data which are the positive and negative energy annihilation operators  $a_\pm(\mathbf{x}), a_\pm^\dagger(\mathbf{x})$  given formally by

$$a_\pm(\chi) = \int \chi^\dagger(\mathbf{x}) a_\pm(\mathbf{x}) d_g^3 \mathbf{x}, \quad (5.18a)$$

$$a_{\pm}^{\dagger}(\chi) = \int a_{\pm}^{\dagger}(\mathbf{x})\chi(\mathbf{x})d_g^3\mathbf{x}, \quad (5.18b)$$

where  $d_g^3\mathbf{x} = \sqrt{g}d^3\mathbf{x}$ . Thus we put

$$\psi(t, \mathbf{x}) = U(t)\pi_+a_+(\mathbf{x}) + U(-t)\pi_-a_-(\mathbf{x}), \quad (5.19)$$

where  $U(\mp t) = \exp(\pm i\omega t)$ . Smearing  $\psi$  by a  $C_0^\infty$  test function  $f(t, \mathbf{x})$  on  $R \times M$ , we get the field

$$\psi(f) = \int f^*(x)\psi(x)d_g^4x, \quad (5.20a)$$

and also by our convention  $\psi^* = \psi^\dagger\gamma^0$ ,

$$\psi^*(f) = \int \psi^*(x)f(x)d_g^4x, \quad (5.20b)$$

with a relation

$$\psi^*(f) = \psi(f)^\dagger. \quad (5.21)$$

Then by (5.19), it is easy to see (5.20) leads to

$$\psi(f) = a_+(f_+) + a_-(f_-), \quad (5.22a)$$

$$\psi^*(f) = a_+^\dagger(f_+) + a_-^\dagger(f_-), \quad (5.22b)$$

where  $f_{\pm}$  on  $M$  are the positive and negative energy components defined by

$$f_{\pm}(\mathbf{x}) = \int e^{\pm i\omega x_0}\pi_{\pm}\gamma^0 f(x_0, \mathbf{x})dx_0. \quad (5.23)$$

We take (5.22) as the precise definitions of  $\psi(f)$  and  $\psi^*(f)$ . With the above preparations, we are now ready to prove the following theorem.

*Theorem 2:* In static space-time, let Dirac field operators be expressed in terms of the creation and annihilation operators on Fermi-Fock space as in (5.22). Then the quantized Dirac field operators satisfy the equation

$$\psi((\mathcal{P} - m)f) = 0, \quad (5.24a)$$

$$\psi^*((\mathcal{P} - m)h) = 0, \quad (5.24b)$$

and the following CAR

$$\{\psi(f), \psi(h)\} = 0, \quad (5.25a)$$

$$\{\psi^*(f), \psi^*(h)\} = 0, \quad (5.25b)$$

$$\{\psi(f), \psi^*(h)\} = -i \langle f, Sh \rangle, \quad (5.25c)$$

together with an integral

$$\langle f, Sh \rangle = \int \int f^*(x)S(x, y)h(y)d_g^4xd_g^4y, \quad (5.26)$$

in terms of a propagator  $S(x, y)$  obtained in §4.

*Proof:* Let  $D = \mathcal{P} - m$ . We can write

$$\psi(Df) = a_+((Df)_+) + a_-((Df)_-). \quad (5.27)$$

By (5.23), it follows by integration by parts

$$(Df)_\pm(\mathbf{x}) = \int e^{\pm i\omega x_0} \pi_\pm \gamma^0 D_\pm f(x_0, \mathbf{x}) dx_0, \quad (5.28)$$

where  $\pi_\pm$  are expressed in (4.21) and

$$D_\pm = \pm \gamma^0 \omega + i \gamma^i \nabla_i - m = \pm 2\omega \gamma^0 \pi_\mp. \quad (5.29)$$

It is then clear to see

$$\pi_\pm \gamma^0 D_\pm = \pm 2\omega \pi_\pm \pi_\mp = 0, \quad (5.30)$$

So we end up with (5.24).

By (5.22) and (5.17a) and (5.17b), it is easy to show (5.25a) and (5.25b). To show the nonvanishing CAR (5.25c), we first compute by using (4.20), (4.23) and (5.23)

$$\begin{aligned} -i < f, S_\pm h > &= -i \int \int S_\pm(x_0 - y_0; \gamma^0 f(x_0, \cdot), h(y_0, \cdot)) dx_0 dy_0 \\ &= \int \int (\gamma^0 f(x_0, \cdot), e^{\mp i\omega(x_0 - y_0)} \pi_\pm \gamma^0 h(y_0, \cdot)) dx_0 dy_0 \\ &= \int \int (e^{\pm i\omega x_0} \pi_\pm \gamma^0 f(x_0, \cdot), e^{\pm i\omega y_0} \pi_\pm \gamma^0 h(y_0, \cdot)) dx_0 dy_0 \\ &= (f_\pm, h_\pm). \end{aligned} \quad (5.31)$$

Then using (5.22), (5.17), (5.31) and (4.26), we get

$$\begin{aligned} \left\{ \psi(f), \psi^*(h) \right\} &= \left\{ a_+(f_+), a_+^\dagger(h_+) \right\} + \left\{ a_-(f_-), a_-^\dagger(h_-) \right\} \\ &= (f_+, h_+) + (f_-, h_-) \\ &= -i < f, S_+ h > -i < f, S_- h > \\ &= -i < f, S h >, \end{aligned} \quad (5.32)$$

which is just (5.25c).  $\square$

If we take the sum

$$f_s = f_+ + f_-, \quad (5.33)$$

then we obtain

$$-i < f, S h > = (f_+, h_+) + (f_-, h_-) = (f_s, h_s). \quad (5.34)$$

We may just take  $\psi(f)$  and  $\psi^*(h)$  as the annihilation and creation operators on Fermi-Fock space  $\mathcal{F}_a(\mathcal{H})$  and define

$$\left\{ \psi(f), \psi^*(h) \right\} = (f_s, h_s). \quad (5.35)$$

This becomes a special case of Dimock's general theorem [5].

Considering Dirac fields as operators, under time evolution we would have by Heisenberg picture

$$\psi(x_0 + t, \mathbf{x}) = e^{iKt} \psi(x_0, \mathbf{x}) e^{-iKt}, \quad (5.36a)$$

$$\psi^*(x_0 + t, \mathbf{x}) = e^{iKt} \psi^*(x_0, \mathbf{x}) e^{-iKt}, \quad (5.36b)$$

where  $K$  is the energy operator. Smearing them by test function  $f(x_0, \mathbf{x})$  it is easy to see the left sides of (5.36) become

$$\int f^*(x_0 - t, \mathbf{x}) \psi(x_0, \mathbf{x}) d_g^4 x = \psi(f(\cdot - t)), \quad (5.37a)$$

$$\int \psi^*(x_0, \mathbf{x}) f(x_0 - t, \mathbf{x}) d_g^4 x = \psi^*(f(\cdot - t)), \quad (5.37b)$$

and the right sides of (5.36) become

$$e^{iKt} \left[ \int f^*(x_0, \mathbf{x}) \psi(x_0, \mathbf{x}) d_g^4 x \right] e^{-iKt} = e^{iKt} \psi(f) e^{-iKt}, \quad (5.38a)$$

$$e^{iKt} \left[ \int \psi^*(x_0, \mathbf{x}) f(x_0, \mathbf{x}) d_g^4 x \right] e^{-iKt} = e^{iKt} \psi^*(f) e^{-iKt}. \quad (5.38b)$$

So we should get

$$\psi_t(f) = \psi(f(\cdot - t)) = e^{iKt} \psi(f) e^{-iKt}, \quad (5.39a)$$

$$\psi_t^*(f) = \psi^*(f(\cdot - t)) = e^{iKt} \psi^*(f) e^{-iKt}. \quad (5.39b)$$

Now we show how this works out.

Define unitary operators for time evolution by

$$U(t) = U^+(t) \otimes U^-(t), \quad (5.40)$$

and

$$U^\pm(t) = \oplus_{n=0}^\infty U_n^\pm(t), \quad (5.41)$$

and

$$U_n^\pm(t) = e^{\mp i\omega t} \otimes \dots \otimes e^{\mp i\omega t}. \quad (5.42)$$

These are all unitary groups with generators  $K$ ,  $K^\pm$  and  $K_n^\pm$  respectively. Here  $K$  is given by

$$K = K^+ \otimes I + I \otimes K^-, \quad (5.43)$$

which is not positive-definite, and  $K^\pm$  is given by

$$K^\pm = \oplus_{n=0}^\infty K_n^\pm, \quad (5.44)$$

and  $K_n^\pm$  is given by

$$\begin{aligned} K_n^\pm &= \omega \otimes I \otimes \dots \otimes I + I \otimes \omega \otimes I \otimes \dots \otimes I \\ &+ \dots + I \otimes \dots \otimes I \otimes \omega \quad (\text{n terms}). \end{aligned} \quad (5.45)$$

We now want to prove (5.39) in terms of the time evolution of creation and annihilation operators by a proposition.

*Proposition 2:* With the above definitions of the Dirac field operators  $\psi(f), \psi^*(f)$  and the energy operators  $K, K^\pm$  and  $K_n^\pm$ , the time evolution of Dirac field operators takes the form (5.39).

*Proof:* By (5.22), this is equivalent to show

$$a(f_\pm(\cdot - t)) = e^{iKt} a(f_\pm) e^{-iKt}, \quad (5.46a)$$

$$a^\dagger(f_\pm(\cdot - t)) = e^{iKt} a^\dagger(f_\pm) e^{-iKt}. \quad (5.46b)$$

By (5.23), we have

$$e^{\pm i\omega t} f_\pm = \int e^{\pm i\omega x'_0} \pi_\pm \gamma^0 f(x'_0 - t, \mathbf{x}) dx'_0 = f_\pm(\cdot - t). \quad (5.47)$$

To show (5.46), it suffices to prove

$$a(e^{\pm i\omega t} f_\pm) = e^{iKt} a(f_\pm) e^{-iKt}, \quad (5.48a)$$

$$a^\dagger(e^{\pm i\omega t} f_\pm) = e^{iKt} a^\dagger(f_\pm) e^{-iKt}. \quad (5.48b)$$

The proof of (5.48) is standard [22].  $\square$

If one replaces the negative energy annihilation operator by a particle conjugated creation operator, then time evolution is implemented with positive energy (refer to [11] for details). This completes our work.

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